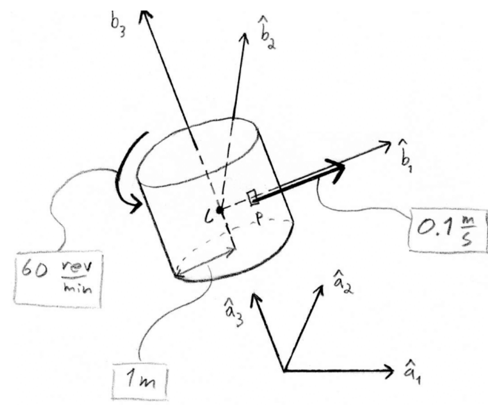
Example Problem:

Problem from lecture 1/15/08,

The mass particle maintains a curved trajectory while it is within spacecraft due to forces being applied from the s/c body. Upon being ejected from the body of the s/c, the mass particle is no longer acted on by any primary forces and thus observes a radial trajectory with respect to the inertial reference frame A.



The equation formally known as "Jim":

$$^{A}\left(\frac{d\vec{r}}{dt}\right) = {}^{B}\left(\frac{d\vec{r}}{dt}\right) + {}^{A}\vec{\omega}^{B} \times \vec{r}$$

Finding the velocity of mass particle P with respect to the inertial frame A,

$$\begin{aligned}
{}^{A}\vec{v}_P &= {}^{A}\left(\frac{d\vec{r}_P}{dt}\right) \\
&= \underbrace{{}^{A}\left(\frac{d\vec{r}_C}{dt}\right)}_{0} + \underbrace{{}^{A}\left(\frac{d\vec{r}_{P/C}}{dt}\right)}_{\text{apply "Jim"}} \\
&= 0 + \left[{}^{B}\left(\frac{d\vec{r}_{P/C}}{dt}\right) + {}^{A}\vec{\omega}^{B} \times \vec{r}_{P/C} \right] \\
&= 0 + \left[0.1\,\hat{b}_1 + 2\pi\,\hat{b}_3 \times 1\,\hat{b}_1 \right] \\
&= 0 + \left[0.1\,\hat{b}_1 + 2\pi\,\hat{b}_2 \right]
\end{aligned}$$

$$\boxed{{}^{A}\vec{v}_P = 0.1\,\hat{b}_1 + 2\pi\,\hat{b}_2}$$

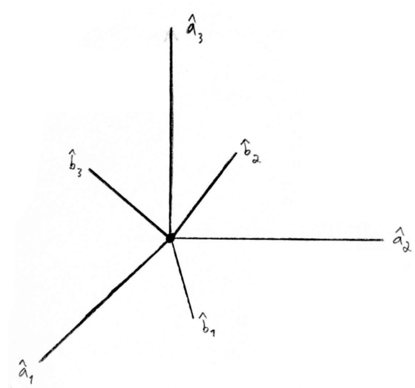
Finding the acceleration of mass particle P with respect to the inertial frame A,

$$
\begin{aligned}
{}^{A}\vec{a}_P &= {}^{A}\!\left(\frac{d^2\vec{r}_P}{dt}\right) \\
&= {}^{A}\!\left(\frac{d^2\vec{r}_C}{dt}\right) + {}^{A}\!\left(\frac{d}{dt}\right) {}^{B}\!\left(\frac{d\vec{r}_{P/C}}{dt}\right) + {}^{A}\!\left(\frac{d}{dt}\right)\left({}^{A}\vec{\omega}^{B} \times \vec{r}_{P/C}\right) \\
&= {}^{A}\vec{a}_C + \left[{}^{B}\!\left(\frac{d^2\vec{r}_{P/C}}{dt}\right) + {}^{A}\vec{\omega}^{B} \times {}^{B}\!\left(\frac{d\vec{r}_{P/C}}{dt}\right)\right] + {}^{B}\!\left(\frac{d}{dt}\right)\left({}^{A}\vec{\omega}^{B} \times \vec{r}_{P/C}\right) + \left({}^{A}\vec{\omega}^{B} \times {}^{A}\vec{\omega}^{B} \times \vec{r}_{P/C}\right) \\
&= {}^{A}\vec{a}_C + {}^{B}\vec{a}_{P/C} + \left(2\,{}^{A}\vec{\omega}^{B} \times {}^{B}\vec{v}_{P/C}\right) + \left({}^{A}\vec{\alpha}^{B} \times \vec{r}_{P/C}\right) + \left({}^{A}\vec{\omega}^{B} \times {}^{A}\vec{\omega}^{B} \times \vec{r}_{P/C}\right) \\
&= 0 + 0 + 2\left(2\pi\,\hat{b}_3 \times 0.1\,\hat{b}_1\right) + 2\pi\,\hat{b}_3 \times 2\pi\,\hat{b}_3 \times 1\,\hat{b}_1
\end{aligned}
$$

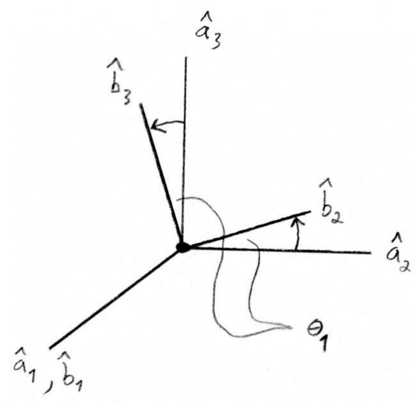
$$
\boxed{{}^{A}\vec{a}_P = \frac{4\pi}{10}\,\hat{b}_2 - 4\pi^2\,\hat{b}_1}
$$

The order of translations and rotations:

When the order of applied translations is altered the resulting position remains unchanged, contrary to this, the order of applied rotations has a direct effect on the resulting special orientation.



Expressing spatial orientation using the angles between reference frames:



$$\hat{b}_1 = \hat{a}_1$$
$$\hat{b}_2 = \sin(\theta_1)\hat{a}_3 + \cos(\theta_1)\hat{a}_2$$
$$\hat{b}_3 = \cos(\theta_1)\hat{a}_3 - \sin(\theta_1)\hat{a}_2$$

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix}}_{\text{Rotation Matrix: } R_1(\theta_1)} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}$$

$$\hat{a}_1 = \hat{b}_1$$
$$\hat{a}_2 = -\sin(\theta_1)\hat{b}_3 + \cos(\theta_1)\hat{b}_2$$
$$\hat{a}_3 = \cos(\theta_1)\hat{b}_3 + \sin(\theta_1)\hat{b}_2$$

$$\begin{aligned} \vec{v} &= x_1\hat{a}_1 + x_2\hat{a}_2 + x_3\hat{a}_3 \\ &= x_1\hat{b}_1 + x_2(-\sin(\theta_1)\hat{b}_3 + \cos(\theta_1)\hat{b}_2) + x_3(\cos(\theta_1)\hat{b}_3 + \sin(\theta_1)\hat{b}_2) \\ &= \underbrace{x_1\hat{b}_1 + (x_2c_1 + x_3s_1)\hat{b}_2 + (-x_2s_1 + x_3c_1)\hat{b}_3}_{\text{where, } s_1 = \sin(\theta_1) \text{ and } c_1 = \cos(\theta_1)} \\ &= \begin{bmatrix} x_1 \\ x_2c_1 + x_3s_1 \\ -x_2s_1 + x_3c_1 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} \end{aligned}$$