AE403: More About Quaternions

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Sequential rotations with quaternions. We have seen that there is a direct correspondence between rotations and quaternions. We can also use quaternions to compute *sequential* rotations. If the quaternion

$$P = ip_1 + jp_2 + kp_3 + p_4$$

represents the orientation of A' with respect to A, and the quaternion

$$Q = iq_1 + jq_2 + kq_3 + q_4$$

represents the orientation of B with respect to A', then the product PQ represents the orientation of B with respect to A. Computing the product PQ is just basic algebra, remembering that $i^2 = -1$, $ij = k$, and so on. When computing this product, however, remember that quaternion multiplication is associative (i.e., $a(bc) = (ab)c$) and distributive (i.e., $a(b+c) = ab + bc$) but not commutative (i.e., $ab \neq ba$).

It turns out there is a trick to compute PQ more quickly. Express the quaternions P and Q each as 4×1 matrices of Euler parameters, so

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$

Then the Euler parameters of the product PQ are given by

$$Q \otimes P = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}.$$

Notice that when we write this product using the notation "$\otimes$", we write P and Q in the same order as for rotation matrices—the first rotation comes last. The 4×4 matrix of q_i's above has a lot of structure. It can be written more compactly as

$$q_4 I + \begin{bmatrix} -S(q) & q \\ -q^T & 0 \end{bmatrix}$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

as usual, and $S(q)$ is a skew-symmetric matrix as we have seen in class.

Relationship between angular velocity and quaternion rates of change. One way to derive this relationship is as follows:

1. Use $\dot{R} = -S(\omega)R$ to show that

$$\begin{aligned}\omega_1 &= \dot{R}_{21}R_{31} + \dot{R}_{22}R_{32} + \dot{R}_{23}R_{33}\\ \omega_2 &= \dot{R}_{31}R_{11} + \dot{R}_{32}R_{12} + \dot{R}_{33}R_{13}\\ \omega_3 &= \dot{R}_{11}R_{21} + \dot{R}_{12}R_{22} + \dot{R}_{13}R_{23}.\end{aligned}$$

2. Plug in expressions for R_{ij} in terms of q_k to find

$$\begin{aligned}\omega_1 &= 2(\dot{q}_1q_4 + \dot{q}_2q_3 - \dot{q}_3q_2 - \dot{q}_4q_1)\\ \omega_2 &= 2(\dot{q}_2q_4 + \dot{q}_3q_1 - \dot{q}_1q_3 - \dot{q}_4q_2)\\ \omega_3 &= 2(\dot{q}_3q_4 + \dot{q}_1q_2 - \dot{q}_2q_1 - \dot{q}_4q_3).\end{aligned}$$

3. Differentiate the constraint $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$ to get one more equation

$$0 = 2(\dot{q}_1q_1 + \dot{q}_2q_2 + \dot{q}_3q_3 + \dot{q}_4q_4).$$

4. Solve this system of four equations to find

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & \omega_3 & -\omega_2 & \omega_1\\ -\omega_3 & 0 & \omega_1 & \omega_2\\ \omega_2 & -\omega_1 & 0 & \omega_3\\ -\omega_1 & -\omega_2 & -\omega_3 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}.$$

We can write this result simply as

$$\dot{Q} = \frac{1}{2}\Omega \otimes Q$$

where the quaternion $\Omega = i\omega_1 + j\omega_2 + k\omega_3$ represents the angular velocity vector (note that Ω is defined with zero real part). Equivalently, you could compute $\dot{Q}$ as the quaternion product $Q\Omega/2$. One more way to write this result is by separating Q into its complex part q and its real part q_4, so we have

$$\begin{aligned}\dot{q} &= \frac{1}{2}(q_4\omega - S(\omega)q)\\ \dot{q}_4 &= -\frac{1}{2}\omega^T q.\end{aligned}$$

Gibbs-Rodrigues parameters. One disadvantage of representing orientation with a quaternion is that it requires four parameters, not three (as for Euler angles). Gibbs-Rodrigues parameters correct this problem. They are defined as

$$g = \begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix} = \begin{bmatrix}q_1/q_4\\ q_2/q_4\\ q_3/q_4\end{bmatrix} = \tan(\mu/2)\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix},$$

where q_1, q_2, q_3, q_4 are the corresponding Euler parameters, and where $\hat{e} = e_1\hat{a}_1 + e_2\hat{a}_2 + e_3\hat{a}_3$ and μ are the corresponding equivalent axis and angle, respectively. The rotation matrix for given Gibbs-Rodrigues parameters g is

$$R = \frac{(1 - g^T g)I + 2gg^T - 2S(g)}{1 + g^T g} = (I - S(g))(I + S(g))^{-1}.$$

The parameters g for a given rotation matrix R are

$$g = \frac{1}{1 + R_{11} + R_{22} + R_{33}} \begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix}.$$

The result g'' of two sequential rotations g and g' can be computed as

$$g'' = \frac{g + g' - S(g')g}{1 - g^T g'}.$$

The kinematic differential equations in terms of Gibbs-Rodrigues parameters are

$$\dot{g} = (I + S(g) + gg^T)\omega/2.$$

Of course, one problem with the Gibbs-Rodrigues parameters is that they have a singularity at $q_4 = 0$, corresponding to $\mu = \pi$. *Modified Gibbs-Rodrigues parameters* handle this problem—they are defined as

$$g_M = \begin{bmatrix} q_1/(1 + q_4) \\ q_2/(1 + q_4) \\ q_3/(1 + q_4) \end{bmatrix}.$$

These parameters have a singularity at $\mu = 2\pi$.