AE403: Active Nutation Control

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In class we derived the response of an axisymmetric spacecraft to a pulsed torque applied about the $\hat{b}_2$ axis. The pulse had magnitude M, started at time τ, and had duration δ. We assumed for convenience (this does not change our final answer) that $\omega_2(0) = 0$. For time $t > \tau + \delta$, in other words after the pulse, the response turned out to be

$$
\begin{aligned}
\omega_1(t) &= \omega_1(0)\cos\lambda t - \frac{M}{\lambda J_t}\cos\lambda(t-\tau) + \frac{M}{\lambda J_t}\cos\lambda(t-(\tau+\delta)) \\
\omega_2(t) &= -\omega_1(0)\sin\lambda t + \frac{M}{\lambda J_t}\sin\lambda(t-\tau) - \frac{M}{\lambda J_t}\sin\lambda(t-(\tau+\delta)).
\end{aligned}
$$

To simplify this result, we apply the trigonometric identities

$$
\begin{aligned}
\cos(a-b) &= \cos b\cos a + \sin b\sin a \\
\sin(a-b) &= \cos b\sin a - \sin b\cos a
\end{aligned}
$$

to rewrite things as

$$
\begin{aligned}
\omega_1(t) &= \left(\omega_1(0) + \frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\right)\cos\lambda t \\
&\quad + \left(\frac{M}{\lambda J_t}\left(-\sin\lambda\tau + \sin\lambda(\tau+\delta)\right)\right)\sin\lambda t \\
\omega_2(t) &= \left(-\omega_1(0) + \frac{M}{\lambda J_t}\left(\cos\lambda\tau - \cos\lambda(\tau+\delta)\right)\right)\sin\lambda t \\
&\quad + \left(\frac{M}{\lambda J_t}\left(-\sin\lambda\tau + \sin\lambda(\tau+\delta)\right)\right)\cos\lambda t.
\end{aligned}
$$

Notice that ω_1 and ω_2 now have a nice form:

$$
\begin{aligned}
\omega_1(t) &= a\cos\lambda t + b\sin\lambda t \\
\omega_2(t) &= -a\sin\lambda t + b\cos\lambda t.
\end{aligned}
$$

As a result, you can verify that

$$\omega_t^2 = \omega_1^2 + \omega_2^2 = a^2 + b^2.$$

So we have

$$
\begin{aligned}
\omega_t^2 &= \left(\omega_1(0) + \frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\right)^2 + \left(\frac{M}{\lambda J_t}\left(-\sin\lambda\tau + \sin\lambda(\tau+\delta)\right)\right)^2 \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\omega_1(0) \\
&\quad + \left(\frac{M}{\lambda J_t}\right)^2 \left(\cos^2\lambda\tau - 2\cos\lambda\tau\cos\lambda(\tau+\delta) + \cos^2\lambda(\tau+\delta)\right) \\
&\quad + \left(\frac{M}{\lambda J_t}\right)^2 \left(\sin^2\lambda\tau - 2\sin\lambda\tau\sin\lambda(\tau+\delta) + \sin^2\lambda(\tau+\delta)\right) \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 \\
&\quad - 2\left(\frac{M}{\lambda J_t}\right)^2 \left(\cos\lambda\tau\cos\lambda(\tau+\delta) + \sin\lambda\tau\sin\lambda(\tau+\delta)\right) \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 - 2\left(\frac{M}{\lambda J_t}\right)^2 \cos\lambda\delta.
\end{aligned}
$$

We will select τ in order to minimize ω_t after a single firing. The minimum will occur when

$$\frac{d\omega_t}{d\tau} = 0.$$

Note that the minimum will also occur when

$$\frac{d\omega_t^2}{d\tau} = 2\omega_t\frac{d\omega_t}{d\tau} = 0.$$

This second expression will be a bit easier to work with. We find that

$$\frac{d\omega_t^2}{d\tau} = 2\frac{M}{\lambda J_t}\left(\lambda\sin\lambda\tau - \lambda\sin\lambda(\tau+\delta)\right)\omega_1(0)$$

so in fact we must have

$$\sin\lambda\tau - \sin\lambda(\tau+\delta) = 0.$$

The most intuitive way to solve this expression is just to draw a sine curve and reason out the different cases for which $\sin a = \sin(a+b)$. For $\lambda\tau < \pi/2$, we must have

$$\lambda(\tau+\delta) - \frac{\pi}{2} = \frac{\pi}{2} - \lambda\tau$$

or rather that

$$\lambda\tau = \frac{\pi}{2} - \frac{\lambda\delta}{2}.$$

Remember we assumed $\omega_2(0) = 0$. So this means that the pulse should be *centered* over the negative peak of ω_2 because for prolate coning the angular velocity is going around backwards. We can compute ω_t after one pulse as follows:

$$\begin{aligned}
\omega_t^2 &= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-\cos\lambda\tau + \cos\lambda(\tau+\delta)\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 - 2\left(\frac{M}{\lambda J_t}\right)^2\cos\lambda\delta \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-\cos\left(\frac{\pi}{2} - \frac{\lambda\delta}{2}\right) + \cos\left(\frac{\pi}{2} + \frac{\lambda\delta}{2}\right)\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 - 2\left(\frac{M}{\lambda J_t}\right)^2\cos\lambda\delta \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-2\sin\frac{\lambda\delta}{2}\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 - 2\left(\frac{M}{\lambda J_t}\right)^2\cos\lambda\delta.
\end{aligned}$$

We apply another trigonometric identity,

$$\sin^2\frac{a}{2} = \frac{1-\cos a}{2},$$

and find that

$$\begin{aligned}
\left(\omega_1(0) - 2\frac{M}{\lambda J_t}\sin\frac{\lambda\delta}{2}\right)^2 &= \omega_1(0)^2 - 4\frac{M}{\lambda J_t}\left(\sin\frac{\lambda\delta}{2}\right)\omega_1(0) + 4\left(\frac{M}{\lambda J_t}\right)^2\sin^2\frac{\lambda\delta}{2} \\
&= \omega_1(0)^2 - 4\frac{M}{\lambda J_t}\left(\sin\frac{\lambda\delta}{2}\right)\omega_1(0) + 4\left(\frac{M}{\lambda J_t}\right)^2\frac{1-\cos\frac{\lambda\delta}{2}}{2} \\
&= \omega_1(0)^2 + 2\frac{M}{\lambda J_t}\left(-2\sin\frac{\lambda\delta}{2}\right)\omega_1(0) + 2\left(\frac{M}{\lambda J_t}\right)^2 - 2\left(\frac{M}{\lambda J_t}\right)^2\cos\frac{\lambda\delta}{2}
\end{aligned}$$

which is the same as what he have above. So we can say that after one pulse,

$$\omega_t^2 = \left(\omega_1(0) - 2\frac{M}{\lambda J_t}\sin\frac{\lambda\delta}{2}\right)^2$$

or rather

$$\omega_t = \omega_1(0) - 2\frac{M}{\lambda J_t}\sin\frac{\lambda\delta}{2}.$$

What we really care about is the change in the transverse angular velocity. Recall that, initially, $\omega_t = \omega_1(0)$ (since $\omega_2(0) = 0$). So we have

$$\Delta\omega_t = -\frac{2M}{\lambda J_t}\sin\frac{\lambda\delta}{2}.$$

Implicitly we assume here that $0 \leq \lambda\delta/2 \leq \pi$. Otherwise we would have $\Delta\omega_t > 0$. Now we turn our attention to the nutation angle. Recall that

$$\tan\theta = \frac{H_t}{H_a} = \frac{J_t\omega_t}{J_a n}$$

so for small angles

$$\theta = \frac{J_t\omega_t}{J_a n}$$

so after one pulse we have

$$\Delta\theta = \frac{J_t \Delta\omega_t}{J_a n} = -\frac{2M}{\lambda J_a n} \sin\frac{\lambda\delta}{2}.$$

What should we pick for δ to maximize $|\Delta\theta|$? We should pick $\lambda\delta = \pi$, or in other words pick the pulse width as one-half the nutation period. This means the optimal strategy is to start thrusting at $t = 0$ and continue for half the nutation period. Note that this (of course) also maximizes $|\Delta\omega_t|$. To summarize, for $\tau = 0$ and $\delta = \pi/\lambda$ we have found that

$$\Delta\omega_t = -\frac{2M}{\lambda J_t}$$
$$\Delta\theta = -\frac{2M}{\lambda J_a n}$$

after a single pulse.