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Control of Many Agents by Moving their Targets: Maintaining Separation

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***Abstract*— Consider a large group of agents chasing a small group of moving targets. Assume each agent moves at constant speed toward the closest target. This paper studies the problem of controlling the agents indirectly by specifying the motion of the targets. In particular, it considers the problem of maintaining a minimum separation distance between each pair of agents, something that is impossible to do with only one target. This paper shows that only two targets are necessary to maintain separation between four agents. It also shows results in simulation to support the conjecture that only two targets are necessary to maintain separation between any number of agents, given suitable initial conditions.**

I. Introduction

This paper is motivated by recent work in which groups of microorganisms are used to manipulate small objects [22]. The microorganisms, of the genus *Paramecium*, are controlled by cyclic application of electric fields. A single paramecium responds to an applied electric field by swimming roughly in the direction from anode to cathode. This response is known as *galvanotaxis* [21]. Given a sufficiently accurate model of its dynamics [18], galvanotaxis can be used as a basis for regulating position or tracking a trajectory with a paramecium [17]. Of course, since all paremecia respond in approximately the same way to an applied electric field, members of a large group can not be controlled individually. Instead, the micro-manipulation presented in [22] relies on the tendency of paramecia to cluster during galvanotaxis. Electric fields are applied to move the center of this cluster in a heuristic circulation pattern ("collide" and "return") in order to push a small object.

A number of questions are raised by this work. For example, is it possible to stretch or shrink the cluster of paramecia along one dimension? Is it possible to maintain two or more stable clusters at the same time? More generally, how can we decide whether an arbitrary configuration of paramecia is reachable, and how can we construct a control policy to achieve any reachable configuration? Is it always necessary to use closed-loop control, or is an open-loop (even sensorless) policy sometimes sufficient? These questions are relevant to a variety of other biological multi-agent systems in addition to paramecia: for example, guiding crowds during emergency egress [15], [19], herding cattle with a robotic sheepdog [24], [25] and invisible fences [7], or interacting productively with cockroaches using a mobile robot [8].

In this paper we consider a simple multi-agent system and focus on the task of maintaining two or more stable clusters of agents. In our model, there are a small number of targets whose trajectory we can specify, and a much larger number of agents that each move at constant speed toward the closest target. We will prove the following two statements: first, that at least two targets are necessary to maintain more than one stable cluster of agents; and second, that only two targets are necessary to maintain four stable clusters of agents. We will also show results in simulation to support the conjecture that only two targets are necessary to maintain any number of stable clusters, given suitable initial conditions.

II. Related work

A. Control of multi-agent systems

Distributed control architectures have been applied successfully to a variety of multi-agent systems, including mobile sensor networks, automated air traffic control systems, and graphical simulations of flocking birds and schooling fish. In each case, an implicit assumption is that we can choose how individual agents respond to each other and to the environment. For example, we might specify that each agent regulates distance from nearest neighbors (to achieve a cohesive swarm) [20], [12], updates its heading based on the average heading of its neighbors (to achieve movement synchronization) [13], or moves toward the circumcenter of its neighbors (to achieve rendezvous) [9].

In this paper, we are interested in multi-agent systems where the dynamics of each agent are fixed. As a result, we can only control agents indirectly by changing external stimuli. This problem has received less attention, mostly in the context of formation control using virtual leaders [14] (in particular when the response of each agent is a linear function of its relative state [23]). Other strategies include commanding the location of a formation's centroid or the variance of its distribution [1], [11]. It is not well understood how to extend these heuristics to more general types of group motion.

B. Pursuit

The questions raised in the introduction are more general than any particular multi-agent system. But because of the system model we consider in this paper, we will be interested here in proving things about the motion of agents that chase moving targets at constant speed. In fact, the paths followed by such agents have been studied for over a century in the mathematics literature, where they are known as *curves of pursuit*. A classic series of papers by Arthur Bernhart provides analytic descriptions of these curves in certain cases, such as when the target is moving along a line or around a circle [2], [3], [5], [4]. Consolidations of this work appear in many textbooks on differential equations, such as [10]. In addition to having obvious military application, pursuit curves have been used for anything from explaining why ant trails are straight [6] to generating formations of mobile robots [16]. Of course, most of this past work assumes that agents are chasing the same targets for all time. In this paper we consider cases in which agents switch from one target to another. Indeed, we will show that this switching is necessary in order to maintain more than two stable clusters of agents.

III. Multi-agent system model

Consider a collection of n agents and m targets. In general, we will assume n is much bigger than m. Let $x_1, \ldots, x_n \in \mathbb{R}^2$ be the position of each agent, and let $z_1, \ldots, z_m \in \mathbb{R}^2$ be the position of each target. All of these variables are functions of time. In our model, each agent moves at constant velocity toward the closest target. For simplicity, we will assume this velocity has unit magnitude. So we can describe the dynamics of each agent i as follows:

$$\frac{dx_i}{dt} = \frac{z_j - x_i}{\|z_j - x_i\|} \quad \text{where} \quad j = \arg\min_k \|z_k - x_i\|.$$

Our goal might be to specify a trajectory $z_j(t)$ for each target $j = 1, \ldots, m$ that achieves a desired trajectory $x_i(t)$ for each agent $i = 1, \ldots, n$. However, it is clear that if $n > m$ then the multi-agent system is not controllable. As a result, there are some agent trajectories than cannot be achieved (for example, try maintaining each agent at a fixed location). Moreover, if $n \gg m$, then the multi-agent system is highly underactuated, so even if there exist target trajectories that result in desired agent trajectories, it is not at all clear how to find them. So instead, we restrict our focus to finding a class of target trajectories that maintains separation between agents.

IV. Separation cannot be maintained using only one target

Assume there is only one target (so $m = 1$). We can show that for any trajectory $z(t)$ of this target, it is in general impossible to maintain separation between pairs of agents. Define the squared distance between any two agents by

$$f(x_i, x_j) = \|x_i - x_j\|^2 = (x_i - x_j)^T (x_i - x_j)$$

First we show that f never increases with time:

$$\begin{aligned}\frac{1}{2}\frac{df}{dt} &= (x_i - x_j)^T \left(\frac{dx_i}{dt} - \frac{dx_j}{dt}\right) \\ &= (x_i - x_j)^T \left(\frac{z - x_i}{\|z - x_i\|} - \frac{z - x_j}{\|z - x_j\|}\right) \\ &= -((z - x_i) - (z - x_j))^T \left(\frac{z - x_i}{\|z - x_i\|} - \frac{z - x_j}{\|z - x_j\|}\right).\end{aligned}$$

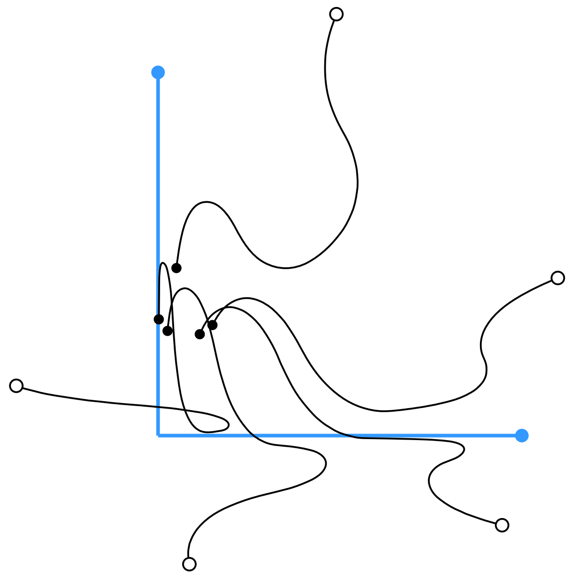

Fig. 1. Five agents following one target that moves out and back along the x-axis and then out and back along the y-axis.

Let $a = z - x_i$ and $b = z - x_j$. Then it remains to show that

$$(a - b)^T \left(\frac{a}{\|a\|} - \frac{b}{\|b\|}\right) \geq 0.$$

But notice that

$$\begin{aligned}(a - b)^T \left(\frac{a}{\|a\|} - \frac{b}{\|b\|}\right) &= \\ \frac{\|a\| + \|b\|}{\|a\|\|b\|}&(\|a\|\|b\| - a^T b).\end{aligned}$$

Then since the Cauchy-Schwarz inequality tells us that $|a^T b| \leq \|a\|\|b\|$, we have

$$(a - b)^T \left(\frac{a}{\|a\|} - \frac{b}{\|b\|}\right) \geq 0$$

as desired. And in fact, we know equality only holds when $a = cb$, or equivalently $z - x_i = c(z - x_j)$, for some scalar $c \in \mathbb{R}$. So we can say that all pairs of agents get closer together (regardless of the target trajectory) unless they are collinear with the target. Under the mild assumption that there exists $\Delta t > 0$ such that for any interval of time $[t, t + \Delta t]$ we can find $\delta > 0$ and $\epsilon < 1$ satisfying

$$|(z - x_i)^T (z - x_j)| \leq \epsilon \|z - x_i\| \|z - x_j\|$$

on a subinterval

$$[t', t' + \delta \cdot \Delta t] \subset [t, t + \Delta t]$$

then $\|x_i - x_j\| \to 0$ as $t \to \infty$ for all i, j. In other words, if the target stays some minimum distance away from the line containing each pair of agents for a nonzero fraction of time, then all agents approach the same location. As an example, consider the target trajectory shown in Fig. 1, in which a group of agents quickly begins to converge.

V. Separation can be maintained between four agents using two targets

Now assume there are two targets (so $m = 2$). We can show that for certain trajectories $z_1(t)$ and $z_2(t)$ of these targets, it is possible to maintain separation between four agents. In particular, let $v > 1$ be the constant speed of each target. Then for any time $t > 0$ we take $s = t \mod (4/v)$ and define

$$z_1(s) = \begin{cases} (vs, 0) & \text{if } 0 \le s \le 1/v \\ (2 - vs, 0) & \text{if } 1/v < s \le 2/v \\ (0, vs - 2) & \text{if } 2/v < s \le 3/v \\ (0, 4 - vs) & \text{if } 3/v < s \le 4/v \end{cases}$$

$$z_2(s) = -z_1(s).$$

So the first target moves from the origin to $(1, 0)$ and back, then from the origin to $(0, 1)$ and back, all at constant speed v. Similarly, the second target moves from the origin to $(-1, 0)$ and back, then from the origin to $(0, -1)$ and back, also at speed v. Both targets continue to repeat these motions for all time.

When this system is simulated, the resulting agent trajectories fall into one of four limit cycles (see Fig. 2). Each limit cycle has a beautiful hourglass shape, and lies entirely in one quadrant of the plane. The limit cycles are also symmetric about the origin, about each axis, and about each $45°$ diagonal. Moreover, these limit cycles are passively stable—target trajectories do not need to be modified in response to perturbations in the trajectory of each agent.

We would like to characterize these limit cycles analytically. Let $\Delta t = 4/v$. For any agent i, we denote the map from $x_i(k\Delta t)$ to $x_i((k+1)\Delta t)$ by $\phi(x)$. We want to show that there are exactly four fixed points of ϕ, one in each quadrant.

A. Interpretation as linear pursuit curves

A *curve of pursuit* is the path taken by an agent that chases a moving target by traveling directly toward it at constant speed. A pursuit curve is called *linear* if the target is moving along a straight line. Notice that the target trajectories we defined above each consist of four straight line segments. As a result, the trajectory of each agent is a sequence of linear pursuit curves.

The shape of a single linear pursuit curve, as shown in Fig. 3, can be described analytically [10]. For the purposes of this derivation, let the target's position be $(0, \eta)$ and the agent's position be (x, y). Assume the target travels at speed v, so $\eta = vt$, and that the agent travels at unit speed, so the arc-length $s = t$. By definition of arc length we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Since we also have

$$\frac{d\eta}{dt} = v\frac{ds}{dt}.$$

then

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{v^2}\left(\frac{d\eta}{dx}\right)^2. \tag{1}$$

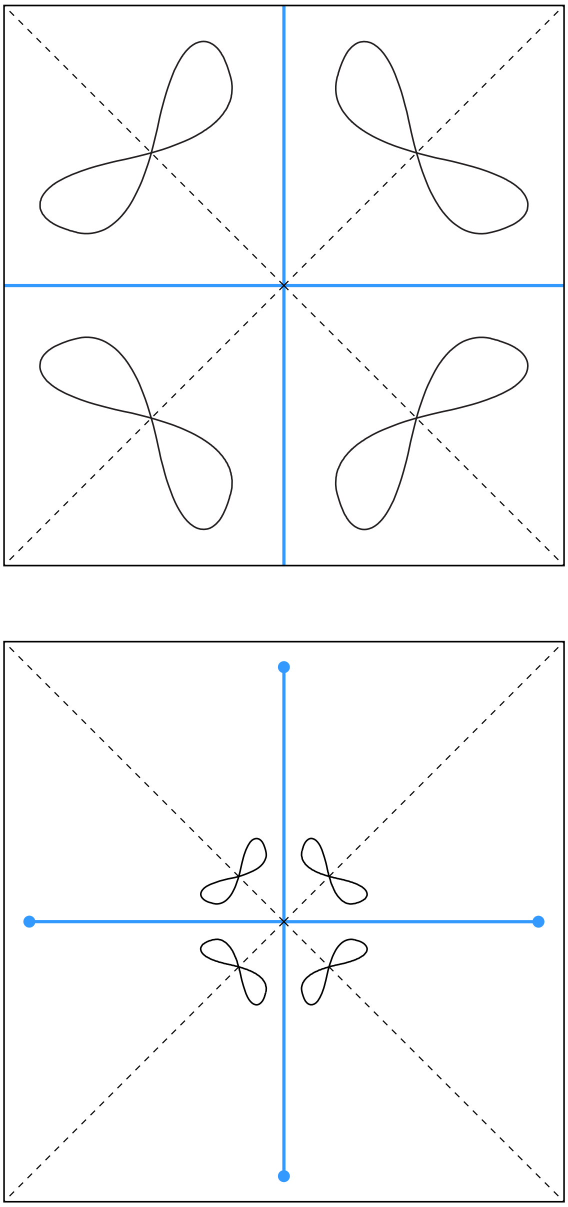

Fig. 2. The agent trajectories resulting from cyclic motion of two targets. These targets repeatedly move out and back, in opposite directions, along first the x-axis and then the y-axis. The top image is a close-up of the bottom image.

The agent moves directly toward the target, so

$$(y - \eta) = \frac{dy}{dx}(x - 0) = x\frac{dy}{dx} \tag{2}$$

Deriving this expression with respect to x we have

$$\frac{d\eta}{dx} = -x\frac{d^2y}{dx^2}$$

Plug this into (1) and we find

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{v^2}\left(-x\frac{d^2y}{dx^2}\right)^2$$

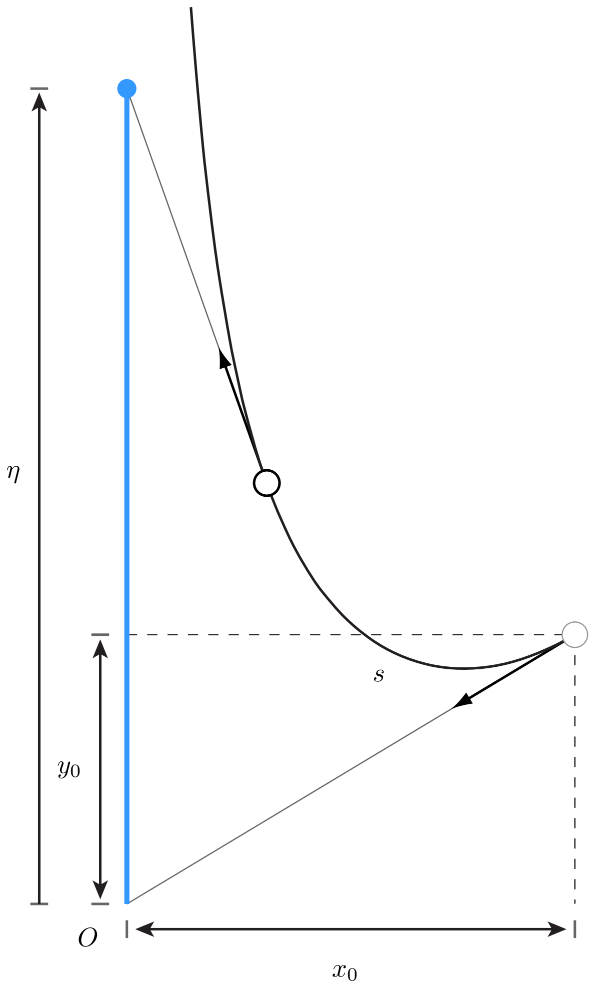

Fig. 3. A curve of linear pursuit. The target starts from the origin and moves along the y-axis at constant speed v. The agent starts from (x_0, y_0) and moves directly toward the target at unit speed.

Let $p = dy/dx$. Then we have

$$1 + p^2 = \frac{1}{v^2}\left(-x\frac{dp}{dx}\right)^2$$

which we can write as

$$\sqrt{1+p^2} = -\frac{x}{v}\frac{dp}{dx}.$$

This expression is integrable:

$$-v\int\frac{dx}{x} = \int\frac{dp}{\sqrt{1+p^2}}$$
$$\Rightarrow \left(\frac{x}{c_2}\right)^{-v} = \frac{p+\sqrt{1+p^2}}{c_1}$$
$$\Rightarrow p = \frac{1}{2}\left(c_1c_2^v x^{-v} - \frac{x^v}{c_1c_2^v}\right).$$

Let $a = c_1c_2^v$. Then we have

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{a}{x^v} - \frac{x^v}{a}\right). \quad (3)$$

Integrating with respect to x, we find the equation for the curve traced out by the follower (assuming that $v \neq 1$):

$$y = \frac{1}{2}\left(\frac{ax^{1-v}}{1-v} - \frac{x^{1+v}}{a(1+v)}\right) + b. \quad (4)$$

The constants a and b are found from initial conditions:

$$a = x_0^{v-1}\left(y_0 - \sqrt{x_0^2+y_0^2}\right)$$
$$b = \frac{v\left(-vy_0 + \sqrt{x_0^2+y_0^2}\right)}{1-v^2}.$$

Combining (2)-(4), we can write an expression for the time taken to reach a particular x and y. It is even possible to invert this expression and write $x(t)$ and $y(t)$ in parametric form. However, the result is not pretty, and how to use it to find fixed points of ϕ analytically is still an open question. Moreover, linear pursuit curves are convex, because

$$\frac{d^2y}{dx^2} = -\frac{vx^{-(v+1)}}{2a}\left(a^2 + (x^v)^2\right)$$

so if $x_0 > 0$ and $y_0 > 0$ then we know $a < 0$ and hence $d^2y/dx^2 > 0$. As a result, bisection on x allows us to find fixed points of ϕ and to compute the shape of associated limit cycles numerically, to any desired precision.

B. *Each quadrant is an invariant set*

Although we have not yet been able to find the fixed points of ϕ analytically, we can show that each quadrant is an invariant set.

First, consider a single linear pursuit curve, where the target moves at constant speed v for a length of time $1/v$. If $x_0 \neq 0$, then (4) implies that $y \to \infty$ as $x \to 0$, so we know that $x(t) \neq 0$ for all $t > 0$. Also, if $y_0 > 0$, then (2) implies that $dy/dx \geq 0$ (meaning that the agent is moving downward) only so long as $y \geq \eta$. As a result, $y(t)$ is bounded below by

$$y(t) \geq \frac{v}{v+1}y_0 > 0.$$

Similarly, if $y_0 < 1$, then (2) implies that $dy/dx \leq 0$ (meaning that the agent is moving upward) for no more than an interval of time $\Delta t = (1-y_0)/v$. As a result, $y(t)$ is bounded above by

$$y(t) \leq y_0 + \frac{1-y_0}{v} < 1$$

Now, without loss of generality, consider an agent that begins in the upper-right quadrant. The agent's trajectory from $0 \leq t \leq 1/v$ can be modeled as a linear pursuit curve for which $x_0 < 0$ and $y_0 > 0$, where we want to verify $x(t) < 0$ and $y(t) > 0$ for $t \leq 1/v$. Similarly, the agent's trajectory from $1/v < t \leq 2/v$ can be modeled as a linear pursuit curve for which $x_0 > 0$ and $y_0 < 1$, where we want to verify $x(t) > 0$ and $y(t) < 1$ for $t \leq 1/v$. But we have just proven both of these results, and the situation for $2/v < t \leq 4/v$ is symmetric. Consequently, we know that the agent remains in the upper-right quadrant for all time. An identical argument shows that each of the other quadrants is also invariant.

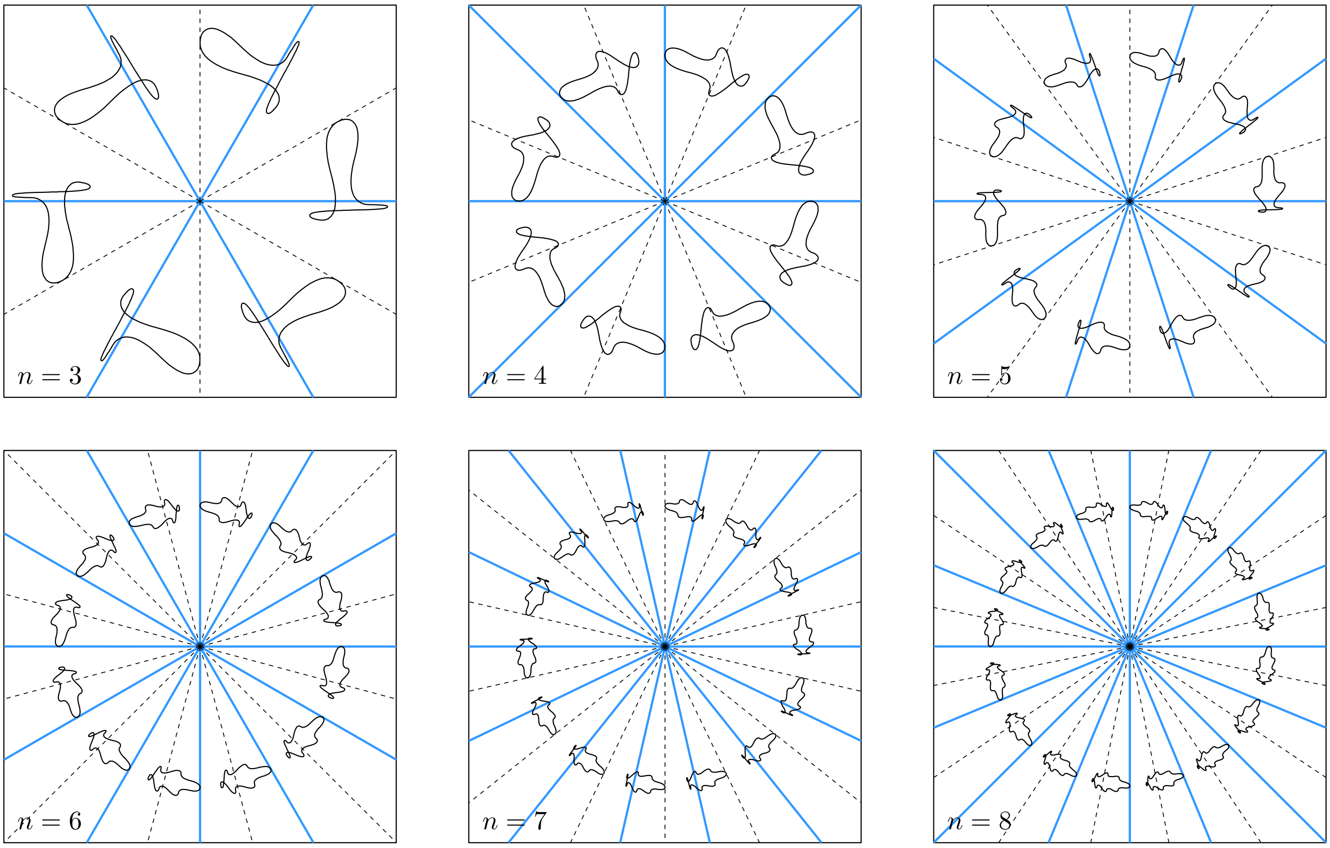

Fig. 4. Agent trajectories resulting from cyclic motion of two targets, each of which repeatedly move out and back along n rotating lines.

VI. Separation can be maintained between any number of agents using two targets

As in the previous section, we assume there are two targets. But now we conjecture that for certain trajectories $z_1(t)$ and $z_2(t)$ of these targets, it is possible to maintain separation between any number of agents (given suitable initial conditions). Although we are still unable to prove or disprove this conjecture, results in simulation lend strong support. In particular, let n be the number of agents and let $v > 1$ be the constant speed of each target. Then for any time $t > 0$ we take $s = t \mod (n/v)$ and define

$$z_1(s) = vs \cdot (\cos(\pi k/n), \sin(\pi k/n))$$
$$z_2(s) = -z_1(s)$$

on each interval

$$\frac{k}{v} < s \leq \frac{k+1}{v}$$

for all $k = 0, \ldots, n-1$. So just as in the previous section, both targets move repeatedly away from and back toward the origin. But now, the targets move along n lines (rather than just 2), rotating an angle π/n between each one.

When this system is simulated, the resulting agent trajectories fall into one of $2n$ limit cycles (see Fig. 4). Each limit cycle lies entirely in a cone, either centered on the target trajectories (when n is odd) or between them (when n is even). These limit cycles show rotational symmetry about the origin. The limit cycles seem to be passively stable as before, but the basins of attraction are not as well-defined. In particular, the cone containing each limit cycle is not an invariant set for $n > 2$ as were the quadrants for $n = 2$ (see Fig. 5 for an example). As a result, if the target speed v is too small, the limit cycles disappear. Note that it is possible to compute the minimum target speed numerically (see Table I).

TABLE I
Minimum Target Speed for Separation

N	v_{min}
2	1
3	9.37
4	17.15
5	30.68
6	41.45
7	61.76
8	75.75

VII. Conclusion

When talking about control of multi-agent systems, we usually assume that the dynamics of individual agents can

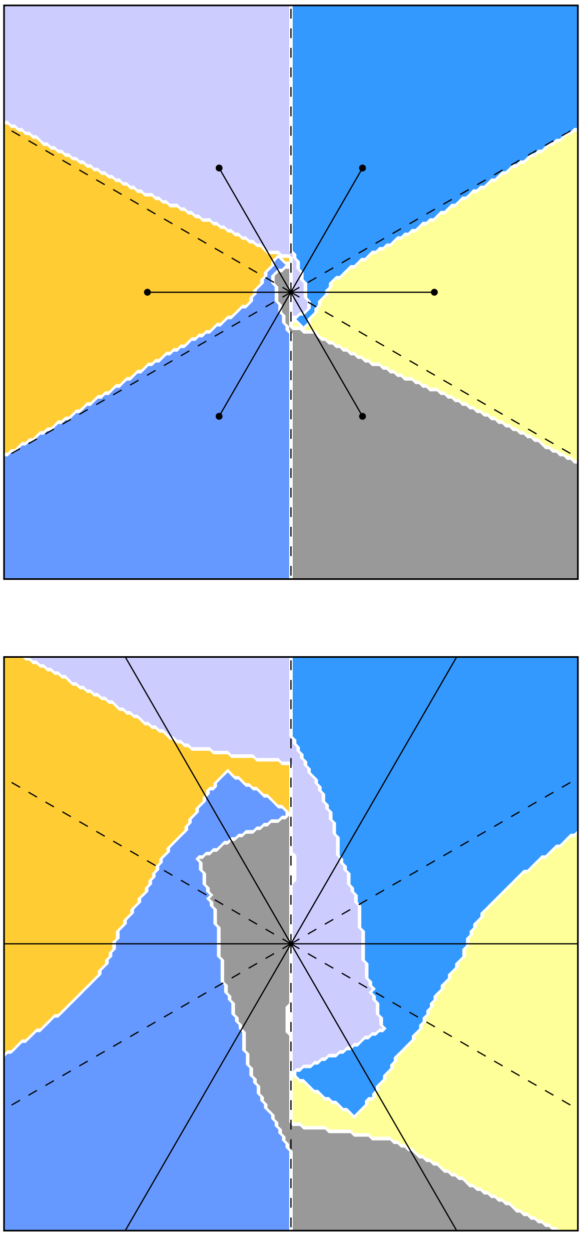

Fig. 5. Invariant sets for $n = 3$, given $v = 9.4$.

be designed. In this paper, we were interested in multi-agent systems where the dynamics of each agent are fixed. Biological systems—such as groups of microorganisms, herds of cattle, or crowds of people—are perfect examples. These systems are controlled indirectly, by applying external stimuli. In general, it is not clear how to plan a sequence of stimuli that cause desired group behavior, nor even how to decide whether a given behavior is achievable. In this paper we considered a simple multi-agent system in which agents chase targets, and focused on the task of maintaining separation between the agents by specifying the target trajectories. We demonstrated in simulation that two targets are sufficient to maintain separation between any number of agents. In future work we hope to address the other questions raised in the introduction, to consider more realistic dynamic models, and to apply our work to actual biological and robotic systems.

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